RELIABILITY IN THE MODELING OF STRUCTURAL DYNAMIC SYSTEMS

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Abstract. The study of uncertainty quantification has been the subject of constant research due to its importance in various fields of engineering especially when it comes to complex designs such as in aircrafts, automobiles, among other examples. The complexity of such projects is revealed when tests of the manufactured systems are carried out and in this way their details can be analyzed and used to reduce uncertainties in the system. On the other hand, mathematical models can be built in appropriate softwares allowing simulations of real situations. Indeed it is possible to make that the model can faithfully represent the real system by quantifying uncertainties and increasing so the reliability of the numerical model built. Taking the example of a beam with fixed - free boundary condition, a comparison between three different systems - designed, modeled and real - will be presented so that the dynamical system is analyzed and its non-parametric uncertainties are calculated. Thus, one can conclude whether the mean model may or may not successfully represent the real model. A positive response will thus allow that the method can be used to increase the reliability of manufactured systems without the need of too long and costly tests.

Keywords: Reliability of Dynamic Models, Modeling Dynamic Systems, Uncertainty in Dynamic Models, Dynamic Models, Uncertainty Quantification.
1 INTRODUCTION

According to Brandão (2007), the uncertainty is related to the variability of the variables that describe the system, and this variability is presented in structural systems in the form of uncertainty. The quantification of uncertainties in this case should be considered especially in models of complex systems, since such systems need to be tested so that details of complexity can be revealed and analyzed, and even then only some information or conclusions can be deduced from them.

However, mechanical mathematical models are constructed so that it is possible to simulate real situations into appropriate software considering that this simulation can replace real experiments that would be performed by systems manufactured from designed systems. In this case one can distinguish three types of models considered in the analysis. They are: (a) designed model, which corresponds to the system designed by design engineers. (b) real model, that refers to the system made from the system designed. In this case, one have differences in geometric parameters, in boundary conditions, materials etc. between the two systems, designed and real. (c) mean model, concerning the modeled system from the designed system, it represents the real system.

The purpose therefore is to make the mean model can faithfully represent the real model, and in this process of modeling the mean model of the designed system, uncertainties are introduced and should be quantified so that it can increase the reliability of the numerical model constructed. It’s very important that all the uncertainties are quantified, but in the case of this article just the epistemic (also called model or nonparametric) uncertainty will be considered. This kind of uncertainties arises mainly due to lack of knowledge of the system, due the errors in the estimation of the theoretical models used in the analysis and not depends on their parameters.

Nevertheless, in this paper one will realized studies comparing the modeled system, called mean model, the 95% confidence limits of the uncertainties propagated in the mass matrix, and the real system that corresponds to the manufactured system. These studies is possible because one have on hand the FRF of mean model, the FRF of stochastic simulations obtained by non-parametric approach, and the FRF of real model obtained by experiments performed with the application of impulse excitation technique. Some variations in the mass matrix also will be induced considering additions of mass in original system in order to bring about changes in the behavior of the response and a consequent increase of uncertainty in the system. In all this studies, different dispersion parameters will be used in each case. Finally conclusions will be showed about the influence of the dispersion parameter in the response of the system, about changes in the response due to the increase of mass in the original system and about the reliability of the mean model.

2 UNCERTAINTIES

The study of uncertainties has been the subject of constant research due to its importance in several areas. According Vandepitte (2011), uncertainty is a potential deficiency in any phase or activity of the modeling process that occurs due to lack of knowledge of the system. It is caused by incomplete information resulting from inaccuracies, lack of specifications (availability of a number of different models that describe the same phenomenon) or disagreement (conflict evident in the description of the phenomenon in question).
It is very important that uncertainty and error not be confused. It can be said that an important difference between these two concepts is mentioned in Huyse and Walters (2001) and corresponds to the fact that errors can be corrected, because it is a deficiency can be recognized, since the uncertainty, because arise due to a lack of knowledge about the system under study, there is no way to be corrected.

On the other hand, it can also be said that while the error is related to the concept of accuracy, which in turn indicates compliance with the true value, uncertainty is related to the concept of precision since it is associated with the repeatability of the analysis or experiment, i.e. relates to the dispersion of the values resulting from the repetition of the same analysis or measurements from tests. It is indicative of the quality and refinement of the results.

An interesting analogy between the concepts of precision, accuracy and the probability distribution of the data considered was proposed by Cabral (2004). In this case the author considered the probability density function (PDF) as the Normal distribution, but in real analysis, the PDF should be determined for each data set analyzed. Therefore, taking into account the Fig. 1 that represents a set of firings of a projectile, it is possible to obtain their respective PDF shown in Fig. 2.

![Figure 1. Cases of precision / imprecision and accuracy / inaccuracy. Source: Cabral (2004).](image1)

![Figure 2. PDFs corresponding to each case in Fig. 1. Source: Cabral (2004).](image2)

In these curves the distance of each peak to the true value is the average error, while the width of each curve, that characterizes the dispersion of the data, represents the uncertainty. One can also say that the area under the curve must be equal, since it is considered that the shooters do the same number of shots. But observing that there are differences in peak
amplitudes, it follows that the lower the dispersion of data greater the peak amplitude of the curve (Cabral, 2004).

In this context one important approach is given in Mattos and Veiga (2002) when study the optimization of the Shannon entropy (some details about this subject can be seen in Justino, 2012) which is given as synonymous with probabilistic uncertainty associated with a probability distribution. According to the authors, "each distribution reflects a degree of uncertainty and different degrees of uncertainty are associated with different distributions (although different distributions may reflect the same degree of uncertainty)." In general it can be said that the more "spread", that is, the higher the dispersion curve, the greater the uncertainty it reflects. Fig. 3 represents the previous statement.

![Figure 3. Representation of uncertainty in different continuous probability distributions. Source: Mattos and Veiga (2002)](image)

Taking into account that uncertainties can be classified in several respects, Junior (2007) gives a brief description of the classification of uncertainties: physical uncertainties, statistical uncertainties, model uncertainties (or modeling uncertainties), uncertainties related to human factors, evaluation uncertainties, phenomenological uncertainties and measurement uncertainties.

The focus of this work is to quantify the uncertainties present in damped linear dynamic structural systems, and, according to Sampaio and Ritto (2008), the uncertainties in these systems can be divided into three types. (a) Uncertainties in loading: are related to the uncertainties of the vibration forces which the system is subjected. (b) Uncertainties in the parameters: those are related with uncertainties in the physical parameters of the system such as geometry and materials. (c) Uncertainties in the model (boundary conditions, beam model, plate models, among others): are related to the approximations made in the mathematical model.

Considering the last two types mentioned, Adhikari (2007), in a generalized way, classifies the uncertainties in dynamic structures into two types. The first one is called parametric uncertainty and relates to the system parameters, such as modulus of elasticity, density, Poisson's ratio, damping ratio and geometric parameters. The second type corresponds to the non-parametric uncertainty (also called epistemic or model uncertainty) that occurs mainly due to lack of knowledge of the system and does not depend on its parameters. As examples may be mentioned errors associated equations of motion, which may be linear or non-linear damping and the model can be considered viscous or non-viscous, among others.
3 IMPORTANCE TO QUANTIFY UNCERTAINTIES

Barrico (2007) mentions that the construction of models that explicitly incorporate uncertainty is important since most real problems cannot be modeled deterministically. This author also comments that what usually happens is that the data associated with problems are incorrect and the quality of solutions is uncertain, then it is necessary to use models that incorporate uncertainty.

Because of this, in recent years one could notice a growing interest in the treatment of uncertainty in real systems. "The model of the physical phenomenon can be idealized as an abstract mathematical representation by a set of equations whose solution seek to replicate analysis and experimental observations" (Sandri, 2010). Nowadays, the idea of translating complex theoretical models for analysis, besides having become an indispensable tool for projects of various structures has been greatly facilitated by the increasing availability of computing resources that also allow the creation of algorithms that for turn provide more refined models. Nevertheless, the mathematical model associated with such systems may include various types of these uncertainties and, according Junior (2007), some of the most important, which can be probabilistically evaluated must be taken into account in a project when seeking to predict the behavior of the structure with the maximum possible accuracy.

In this context, the calculation of the uncertainties should not be left out of the project, since it makes them more reliable. According to Cabral (2004) "with the indication of a realistic estimate of the uncertainty, the information contained in the results becomes much more useful."

According Oberkampf (2001) random uncertainties have its mathematical representation by probability distributions wherein "the propagation of these distributions by a process of modeling and simulation is well developed and is described in many texts. "With respect to epistemic uncertainty such mathematical representation is a challenge. Furthermore, the author also states that the parametric uncertainties can be entirely random in nature, and the model uncertainties are fundamentally epistemic in nature" because it relates to real structural changes in the model."

Otherwise it can be said that when one develop a project of a mechanical system, the intention is to build it so that it is at the highest level of accordance with the proposed project. However, especially in cases of complex projects, such as aircraft, spacecraft, automobiles, among others, the production is never identically to the original design, it has errors and uncertainties. The fact that the produced model not represents the designed model exactly as it occurs because are made approximations and simplifications in the model beyond what some details are unknown or are not known with precision.

In the case of model uncertainties, and intending to increasing reliability of the manufactured component, they must be taken into account to improve the predictability of the model produced. The first step to this is to build a model in appropriate software that corresponds to the manufactured component and which will be called mean model. The confidence interval for the mean model will be constructed and, with that, the manufacturer can know the limits in which the manufacture of the mean model, which as stated earlier corresponds to the manufactured component, must occur so that it is within the specifications required by company in question.

As defined in Junior (2007), mathematical models are basic tools for engineering calculations, are idealizations and it describe reality within a certain degree of approximation,
which is not fully known. A model implies in a simplified representation of the structural
system, which makes it possible to obtain mathematical expressions capable of describing the
behavior of the system with sufficient accuracy.

Therefore, in a generalized manner one said that the model uncertainty may be
understood as the deviation between the actual system considered and its simplified
representation, especially when dealing with complex systems, in which the translation to
models with theoretical assumptions are never in perfect agreement with the real system to be
analyzed.

In conclusion, it is important to note that due to so many requirements on projects of
dynamical systems, and also due to the complexity of most of these projects, the uncertainties
should be taken into account and calculated that, with this, it has greater reliability on the
numerical models, which also increases its credibility, since uncertainty quantification plays a
key role in this case. Furthermore one has nowadays advanced computational resources that
provide important support to implementation of methods and approaches to uncertainty
quantification.

4 CHARACTERISTICS OF THE DYNAMIC SYSTEM UNDER STUDY

In this paper will be analyzed a damped linear dynamic structural system with \( n \) degrees
of freedom. It will be represented by a steel beam with fixed-free boundary conditions. Its
dimensions, mass and properties are:

- length: 200.000 mm; height: 23.064 mm; thickness: 2.736 mm; mass: 137.28 g;
  modulus of elasticity: 183.000 GPa; density: 7.830 g/cm³.

The equation of motion of this system can be represented in the frequency domain as
follows:

\[
-\omega^2 \mathbf{M} \ddot{\mathbf{u}}(\omega) + i\omega \mathbf{C} \dot{\mathbf{u}}(\omega) + \mathbf{K} \mathbf{u}(\omega) = \mathbf{f}(\omega)
\]

in which the vectors \( \mathbf{u}(\omega) \), \( \dot{\mathbf{u}}(\omega) \), \( \ddot{\mathbf{u}}(\omega) \) respectively represent the vector of displacement,
velocity and acceleration of the mass. \( \mathbf{f}(\omega) \) is the vector of the external force applied to the
system. All of these symbols are frequency dependent.

\( \mathbf{M} \), \( \mathbf{C} \) and \( \mathbf{K} \) are respectively
the \( n \times n \) random matrices of mass, damping and stiffness. Such matrices are real, symmetric
and positive-defined; they belong to positive-definite ensemble proposed by Soize (2000) and
studied in Soize (2003a) and Soize (2005a), whose details can be found in Justino (2012).

The features of this system can be summarized as following.

4.1 Frequency band

The frequency band for this system is given by:

\[
\mathcal{B} = [\omega_{\text{min}}, \omega_{\text{max}}], \quad 0 < \omega_{\text{min}} < \omega_{\text{max}}
\]

in which \( \omega_{\text{min}} \) is the minimum frequency and \( \omega_{\text{max}} \) corresponds to the maximum frequency.

For the case of this study, one has:

\[
\mathcal{B} = [0,1200] \quad [\text{Hz}]
\]
4.2 Response of the dynamic system

The response of a damped linear dynamic structural system on the frequency domain is:

\[ H(\omega) = (-\omega^2 M + i\omega C + K)^{-1} \]  

(4)

where \( H(\omega) \) is the frequency response function (FRF) of the studied system obtained for a defined frequency band.

4.3 Excitation force

The external force vector can be represented by:

\[ \mathbf{f} = (f_1, f_2, \ldots, f_n)^T \]  

(5)

For the case of this study, one has:

\[ \mathbf{f} = (0, \ldots, f_{79}, \ldots, 0)^T \]  

(6)

4.4 Results

The signal obtained on the analysis is a velocity signal, then:

\[ \dot{\mathbf{u}} = (\dot{u}_3, \dot{u}_2, \ldots, \dot{u}_n)^T \]  

(7)

For the case of this study, one has:

\[ \dot{\mathbf{u}} = (0, \ldots, \dot{u}_{159}, \ldots, 0)^T \]  

(8)

5 DESIGNED, MODELED AND REAL SYSTEMS

In this work, three types of systems will be considered, the designed, modeled and real systems. They will be described following.

5.1 Designed system

The designed system corresponds in this case to the mechanical system conceived by designers. In this process is defined the material, geometrical parameters among others. The designers can project a simple system, like for example, a beam or a complex system such an aircraft or a vehicle.

5.2 Real system

The real system is the manufactured system produced from designed system. In this case the manufactured system is made by men, then the geometrical parameters, the boundary conditions, the material, mass, density, Poisson ratio among others parameters does not exactly coincide with the parameters of the designed system. Consequently, the real system has then to be considered as an uncertain system with respect to the designed system.

5.3 Modeled system

The mean model corresponds to predict model. According to Soize (2005a), “[…] the predictive model will be developed to predict the static displacement field of a static system
subjected to a given external static load” for example. Furthermore, according to the same author, this predictive model “[…] is constructed by developed a mathematical mechanical model of the designed system for a given input”.

In the case of this paper, the predicted model or the mean model was represented by a damped linear dynamic structural system. The mean matrix of mass and stiffness is obtained by Finite Element Method. The dynamic system constituted by a steel beam was modeled with boundary conditions fixed-free, 80 elements of Euler - Bernoulli on a model with 81 nodes and 160 degrees of freedom.

The damped matrix was considered as following:

\[
\tilde{C} = \text{coef}_1 \times \tilde{M} + \text{coef}_2 \times \tilde{K}
\]

(9)

where

\[
\text{coef}_1 = 1 \quad \text{and} \quad \text{coef}_2 = 10^{-7}
\]

(10)

The FRF of the mean model is:

\[
-\omega^2 \tilde{M} \ddot{\bar{u}}(\omega) + i\omega \tilde{C} \dot{\bar{u}}(\omega) + \tilde{K} \bar{u}(\omega) = \tilde{f}(\omega)
\]

(11)

\[
\bar{H}(\omega) = (-\omega^2 \tilde{M} + i\omega \tilde{C} + \tilde{K})^{-1}
\]

(12)

The bars under the symbols represent mean values. The vectors and the matrices are deterministic. The mass, damped and stiffness matrices are real, symmetric and positive-definite matrices.

The FRF of the mean model is obtained to be analyzed with the FRF of the experiment and the FRF of the stochastic simulation of the same system. These responses will be showed on section 10.

6 STOCHASTIC MODELING

The stochastic modeling is the first step in order to proceed to the uncertainty quantification. First, one must choose the matrices that will be randomizes. In case of this procedure the mass, damping and stiffness matrices will be called \( G \) in order to make a general procedure. This may be considered because, according to Adhikari (2007), the random matrices of the dynamic system have similar probabilistic characteristics. Thereafter, the sample space must be defined. It identifies the values that can be assumed by the random matrices and corresponds to the construction of the probability density function (PDF) for each of the matrices considered earlier. It should also say that at this stage of modeling, the success of the process depends of use of the appropriate PDF for each of the random matrices, so that are eliminated on the analysis errors resulting from the use of an incorrect PDF. Fortunately, in this case one have the Wishart distribution obtained by non-parametric approach.

The nonparametric approach uses the Random Matrix Theory (RMT) and de Maximum Entropy Principle (MEP). This theory was proposed by Soize (1998) and Soize (2000), its validation was realizes on Soize (2001), Soize (2003a), Soize (2003b) e Chebli et al (2004), and an overview about the approach was made by Soize (2005b). Some details about the nonparametric approach, including RMT and MEP, can be seen in Justino (2012).

The Wishart distribution was studied in more detail by Adhikari (2007), Adhikari (2008) and Adhikari (2009). Some details of this studies cam also be seen in Justino (2012).
In addition, the modeling of uncertainty includes vibration problems in the range from low to high frequency. But important information should be remembered here, the parametric and nonparametric approaches should be considered taking into account. Besides the type of uncertainty that need to be quantified, also the vibration frequency range needs to be considered for the analyses. In case of low frequency bands of vibration, parametric uncertainties are considered in detail. Already in mid frequency ranges, parametric and nonparametric uncertainties should be quantified. Finally, as regards the high-frequency bands of vibration, the nonparametric uncertainties need to be quantified.

Details about how to obtain the Wishart distribution for uncertainty quantification in structural dynamics models isn’t the subject on this article, but it can be seen on the references cited above.

However it’s of great importance to know that the Wishart distribution is represented by the follow equation with its optimum parameters by:

\[
p_{\mathcal{G}}( \mathbf{G} ) = \left( 2 \pi \right)^{n_p} \Gamma_n \left( \frac{1}{2} \right) \left| \Sigma \right|^{\frac{1}{2}} | \mathbf{G} |^{\frac{1}{2} (p-n-1)} \exp \left\{ \operatorname{tr} \left( -\frac{1}{2} \Sigma^{-1} \mathbf{G} \right) \right\}
\]

with parameters

\[
p = \theta + n + 1
\]

\[
\Sigma = \frac{\theta}{\theta}
\]

in which

\[
\theta = \frac{1}{\delta^2} \left\{ 1 + \frac{\left( \operatorname{tr}( \mathbf{G} ) \right)^2}{\operatorname{tr}( \mathbf{G} )} \right\} - (n + 1)
\]

in which \( p_{\mathcal{G}}( \mathbf{Z} ) \) is the Wishart PDF of \( \mathbf{G} \), \( \Gamma_n \) is the Gamma Function \( p \), is a scalar parameter of Wishart PDF and \( \Sigma \) is a matrix parameter of Wishart PDF.

7 STOCHASTIC SIMULATION

"Simulation is a process of reproduction of the real world based on a set of hypotheses and real designed models" (Castanheira (2004) apud Ang and Tang (1984)). The Monte Carlo method is used to be able to reproduce the behavior of the system under study, perform simulations and obtain statistical response for analysis of results.

It is necessary to generate a sufficient number of samples so that one can obtain statistics of the response, or to determine a number of moments (mean and dispersion). The main problem now is to determine how many simulations are needed to construct an approximation of the response to pre-defined error. For this, we used the method of quadratic convergence.

According to Soize (2005b), the convergence in accordance with the size of the random matrix and the number of realizations required in Monte Carlo simulation, is given by:

\[
\text{conv}(n_s, n) = \left\{ \frac{1}{n_s} \sum_{k=1}^{n_s} \int_{\omega \in \mathbb{B}} \left\| Q^n(\omega; \theta_k) \right\|^2 d\omega \right\}^{1/2}
\]

in which \( n \) is the order of random matrices; \( n_s \) corresponds to the number of Monte Carlo simulation, \( \omega \) is the frequency on band \( \mathbb{B} \), \( Q^n(\omega; \theta_k) \) corresponds to the response of the stochastic system calculated for each simulation \( k \) with corresponding result \( \theta_k \).

The simulations are repeated until a convergence criterion is confirmed. A deviation value needs to be considered for this convergence.
In case of this paper, this stochastic simulation was made by a computational program develop by the author. It was performed for a dynamic system described on section 4. The propagation of uncertainty occurs in the random matrix of mass. 95% confidence limits are obtained in this case.

The results of these simulations will be showed and analyzed with the FRF of the mean model and the FRF of the experiment of the same system and it will be showed on section 10.

8 DISPERSION PARAMETER

The dispersion parameter $\delta$ is the information about the uncertainty on the system. It’s very important information for uncertainty quantification.

The dispersion parameter must be calculated within an interval of possible values considering the nonparametric approach. According to Soize (2003a) the equation that must be used for the calculation of this parameter is given by:

$$0 < \delta < \left(\frac{n_0+1}{n_0+5}\right)^{1/2}$$

in which $n_0 > 1$ is an integer that is given and fixed, and in this case, $\delta$ is independent of $n$. It can be stated that, in general, the dimensions of the model in question are large and sometimes above 100. If an example, if consider $n = 10$ then will be $n_0 = 10$ which, when applied in Eq. (12) correspond to a high uncertainty (0.856) which generally is not achieved in applications (Soize, 2003a).

In case of this paper will be used the values of:

$$\delta_{1M} = 0.11$$
$$\delta_{2M} = 0.20$$

wherein $\delta_{1M}$ and $\delta_{2M}$ are dispersion parameters of the random matrix of mass considered in the analysis.

9 EXPERIMENT

In this section is shown details about the experiment performed with a dynamic system constituted by a steel beam like explained on section two.

The tests were performed at the Mechanical Vibrations Laboratory of Mechanical Engineering Institute, Federal University of Engineering (UNIFEI) – Itajubá – and it aims at obtaining the FRF of the dynamic system tested, which in this case is represented by the steel beam already described in section 4.

9.1 Configuration of experiment

The experiment was performed for the frequency range 0-1200 Hz and the propagation of uncertainty occurs in the random matrix of mass. In addition to test of the beam, two other configurations were tested. These configurations consist of the addition of masses (represented by magnets) in the beam. These variations in the mass were performed in order to demonstrate that any change in the project, or in this case, on the beam under study, be it an increase in reinforcing or fixing equipment in the beam for instance gives rise to even more
uncertainty that cause changes on the system response. The FRF in this case becomes different of the FRF of original design.

The configurations are given by:

Configuration 1: beam without the addition of magnets showed in Fig. 4a.

Configuration 2: beam with addition of two magnets at positions 95 mm and 150 mm from the beam showed in Fig. 4b.

Configuration 3: beam with addition of four magnets at positions 20 mm, 60 mm, 95 mm and 150 mm from the beam showed in Fig. 4c.

Measurements were taken from the left end to the right end of the beam.

![Configuration of experiment. (a) Configuration 1. (b) Configuration 2. (c) Configuration 3.](image)

The impulse was caused by an instrumented impact hammer and the excitation point was in the central position of the beam corresponding to the length of 100 mm from their ends. Already reading results, represented by the speed signal caused by the movement of the beam is picked up by the laser vibrometer at its end that corresponds at the position 200 mm in length. The signals from the load cell of the impact hammer and of laser vibrometer is captured to an analyzer of signal which then shows the curves for the FRF obtained in the testing. A general configuration of the experiment can be seen in Fig. 2.

10 SIMULATION AND EXPERIMENT RESULTS

This section presents the results (FRF) obtained in stochastic simulations that corresponds to 95% confidence limits for uncertainty, the FRF of the mean model and the
FRF obtained by the experiment. The frequency band of 0 – 1200 Hz was divided into two frequency bands: 0 – 800 Hz and 800 – 1200 Hz. Then was considered the two dispersion parameters shown on Eq. (16) for each of the two frequency band defined. The results are shown as following.

10.1 Frequency band: 0 – 800 Hz

Dispersion parameter = 0.11.

![Figure 6](image)

Figure 6. (a) Convergence. (b) 95% confidence limits. $\delta_K = 0.11$. Without additional magnets.

![Figure 7](image)

Figure 7. (a) Convergence and (b) 95% confidence limits. $\delta_K = 0.11$. With addition of two magnets.
Observing Fig. 6a, Fig. 7a and Fig. 8a, it can be said that a good convergence is reached to a value of number of simulations $n_x = 1000$.

Considering now the Fig. 7b and Fig. 8b, there was no agreement between the FRF of the mean model and the FRF of the experimental test and these two curves are in large part outside of the 95% confidence limits. Moreover, when observing Fig. 6b, although there is disagreement of the FRF also in the frequency range 0-200 Hz due the gain to the experimental FRF, it can be said that the FRF of the mean model and the FRF of the test agreed well for a frequency range approximately 200 to 450 Hz, remaining within the 95% confidence limits calculated.

$Dispersion\ parameter = 0.20$. 

Figure 9. (a) Convergence and (b) 95% confidence limits. $\delta_K = 0, 2$. Without additional magnets.
Figure 10. (a) Convergence and (b) 95% confidence limits. \( \delta_K = 0, 2 \). With addition of two magnets.

Figure 11. (a) Convergence and (b) 95% confidence limits. \( \delta_K = 0, 2 \). With addition of four magnets.

By observing the Fig. 9a, Fig. 10a and Fig. 11a, one can say that as in the previous case, good convergence is attained for a value of number of simulations \( n_s = 1000 \).

Taking into account the Fig. 10b and Fig. 11b, there is no agreement between the FRF of the mean model and the FRF of the test. On the other hand, in the case of Fig. 9b, although there is a disagreement of the FRF also in the frequency range 0-200 Hz due the gain to the experimental FRF, the FRF of the mean model and the FRF of the experimental test continued agreeing for frequency range approximately 200-450 Hz, but one can see the output of these curves of 95% confidence limits calculated.
10.2 Frequency band: 800 – 1200 Hz

Dispersion parameter $\delta_K = 0.11$.

Figure 12. (a) Convergence. (b) 95% confidence limits. $\delta_K = 0.11$. Without additional magnets.

Figure 13. (a) Convergence and (b) 95% confidence limits. $\delta_K = 0.11$. With addition of two magnets.

Figure 14. (a) Convergence and (b) 95% confidence limits. $\delta_K = 0.11$. With addition of four magnets.
By observing the Fig. 12a, Fig. 13a and Fig. 14a, one can say that a good convergence is attained for a value of number of simulations $n_s = 1000$.

When verifying Fig. 13b and Fig. 14b, it is possible to say that there is no correlation between the FRF of the mean model and the FRF obtained in the test for the frequency range showed. One can also observe that these results remain partly within the 95% confidence limits obtained by stochastic simulation. However, there is a perceived closeness of the curves FRF of the mean model and the FRF of the test for frequencies above approximately from 980 Hz to 1200 Hz in Fig. 12b, and the FRF of the mean model remains entirely within the 95% confidence limits while the FRF of the test lies large partly within this same range.

**Dispersion parameter** = 0.20.

![Figure 15.](image1.png)

Figure 15. (a) Convergence and (b) 95% confidence limits. $\delta_k = 0, 2$. Without additional magnets.

![Figure 16.](image2.png)

Figure 16. (a) Convergence and (b) 95% confidence limits. $\delta_k = 0, 2$. With addition of two magnets.
When analyzing Fig. 15a, Fig. 16a and Fig. 17a, it is clear that the value of the number of simulations suitable for this case corresponds to $n_s = 1000$.

Analyzing Fig. 15b, Fig. 16b and Fig. 17b, one can verify that in the frequency range below 980 Hz the results are not satisfactory, which occurs also in Fig. 16b and Fig. 17b for the frequency range above 980 Hz. Verifying the Fig. 15b one find a greater closeness of curves of FRF of the medium model and the FRF of the test for frequencies approximately from 980 Hz to 1200 Hz. Moreover, in this same range, the curves of FRF are inserted in a 95% obtained in the simulation.

11 CONCLUSIONS

The first conclusion of this study is about the results of the convergence. When one analyze the convergence curves, there are a good convergence to a value around $n_s = 1000$ of Monte Carlo simulations, regardless of dispersion parameter used, which means that this amount is sufficient so that one can obtain reliable results for the system under study.

In case of Fig. 6b, relating to obtaining a 95% confidence limits for $\delta_K = 0.21$, frequency range 0-800 Hz and without addition of magnets, it could be seen a good agreement between the results of the FRF of the mean model and the experimental FRF for a frequency range approximately 200-450 Hz. This result is not seen in Fig. 7b and Fig. 8b, which were obtained under the same conditions of the results presented in Fig. 6b, but in these cases with the addition of two and four magnets respectively. Then, in a general way it can be said through the observation of the results obtained that the mean model does not represent well the real system tested in none of the settings except for the frequency range 200-450 Hz in Fig. 6b. This is justified by the fact that at low frequencies the nonparametric quantification of uncertainty is not sufficient to adjust the mean model to the reality of the studied system. In fact, for low frequency bands, the quantification of parameter uncertainty is predominant.

In case of Fig. 9b, Fig. 10b, Fig. 11b relating to obtaining a 95% confidence limits for $\delta_K = 0.20$, frequency range 0-800 Hz can be concluded from the results presented here that, in general, the FRF of the mean model obtained does not represent the real system in none of configurations. The justification for this result is the same as the previous paragraph, at low frequencies is necessary a parametric analysis. With regard to the two curves of FRF (mean model and experiment), they remain outside of the 95% IC calculated. This fact can be
justified by the value of the dispersion parameter be high to such a study, since the simulated system corresponds to a simple structure.

In case of Fig. 13b and Fig. 14b considered for frequency range 800 - 1200 Hz, $\delta_K = 0.11$ and without addition of magnets, the mean model do not represent well the real system tested. By the way, analyzing Fig. 12b, it can be observed that there was a good proximity between the FRF of the mean model and the FRF of the test for a frequency range approximately from 980 Hz to 1200 Hz. Beyond what these curves are inserted in the 95% confidence limits. The result of Fig 12b is an example in which it is found that in high-frequency bands the quantification of non-parametric uncertainty is predominant.

In case of Fig. 13b and Fig. 14b considered for frequency range 800 - 1200 Hz, $\delta_K = 0.20$ and without addition of magnets, in general, one can say that the FRF of the mean model satisfactorily represents the real system for the frequency range 980 - 1200 Hz when considering Fig. 15b, which does not occur in other results in Fig. 16b and Fig. 17b. The justification here is the same as the previous paragraph, that is, quantifying the nonparametric uncertainty becomes increasingly prevalent with increasing frequency range analyzed. With regard to the two curves of FRF (mean model and experiment), they remain partially outside of the 95% IC for the frequency range 800-980. This fact can be justified by the value of the dispersion parameter be high to such a study, since the simulated system corresponds to a simple structure.

However, it can be observed that for the first configuration tested (without adding magnets) and independent of the dispersion parameter and the frequency range, the results were satisfactory for at least a small frequency band, which did not occur with other settings. What one wants to show with this study is that the response of the analyzed problem changes when adding more mass to the initial system, as this causes an increase of model uncertainty that was not considered in the original design of that system.

On the other hand, some observations about the behavior of the response need to be done in view of different dispersion parameters considered in the analysis. Regarding the results obtained for the FRF of the mean model, FRF of the test and a 95% confidence limits in the frequency range 0-1200 Hz and without adding mass, which corresponds to the real system – Fig. 6b, Fig. 9b, Fig. 12b e Fig. 15b, and taking as reference the vertical axis of these graphs, it can be seen that the confidence region increases as the dispersion parameter values become higher. One can observe in the same graph results, but now with reference to the horizontal axis that as the frequency increases the width of the 95% confidence limits also increases on the natural frequencies region. This means that the uncertainty in the system increases with increasing frequency thus hampering its predictability.

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**REFERENCES**


