CONNECTIONS BETWEEN SCREW THEORY AND CARTAN’S CONNECTIONS

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Abstract—— In this paper some relations that exist between expressions in Screw Theory of Robotics and Differential Geometry, from the point of view of Cartan, are presented. The ideas were inspired in books of Differential Geometry, Lie groups and Lie algebras and its applications. The Cartan’s connection is the initial concept, followed by the covariant derivative. It is presented how kinematics and dynamics are obtained, in case of pure rotations and general movement in Euclidean space. The generalization of these concepts to Riemannian manifolds are then presented, and some conclusions are drawn concerning the possibility of kinematic chain calculations in such space (propagation of velocities). Finally, the idea of using connections and covariant derivatives are extended to multi-frame cases (as is common in Robotics) in Euclidean space.

Keywords—— Screw Theory, Riemannian geometry, Cartan’s Connection.

Resumo—— Este artigo apresenta algumas relações existentes entre expressões e formulações em Robótica (em particular na Teoria das Helicóides) com Geometria Diferencial do ponto de vista de Cartan. A ideia foi inspirada em livros de Geometria Diferencial e suas aplicações. A conexão de Cartan é o ponto de partida para a interpretação posterior, assim como a derivada covariante. Apresenta-se como a cinemática e a dinâmica de sistemas de corpos rígidos, com especial interesse para a área de modelagem de sistemas robóticos, são deduzidas, inicialmente no caso de rotações puras, e depois para o caso de movimento geral no espaço Euclidiano. Os conceitos são generalizados para variedades Riemannianas, e conclusões importantes sobre Robótica são tiradas para estes espaços (no que se refere a cadeias cinemáticas). Finalmente, o uso de conexões de Cartan para sistemas com muitos frames de referências (pelo menos um inercial) é apresentado.

Palavras-chave—— Teoria das Helicóides, Geometria Riemanniana, Conexão de Cartan.

1 Introduction

Concepts of Differential Geometry and Lie Groups and Algebras have been used in Robotics for a long time, with the first appearing in classical references as (Craig, 1989), where transformation between two reference frames are matrices in the matrix group $SE(3)$ (or $SE(2)$, if the manipulator’s movement is planar). More recently, Screw Theory, that was developed by (Ball, 1876), started to be applied, but the modern definitions of screws, twists and wrenches as elements of $SE(3)$, $se(3)$ and $se^*(3)$, (respectively, a Lie group, its Lie algebra and Lie co-algebra), are far more recent (Sattinger and Weaver, 1986). Today, there are very complete references on the subject (Murray et al., 1994).

Cartan’s connections are geometrical objects that are fundamental to modern Geometry and Physics, besides the concept of moving frames (Schwarz, 1996), (Choquet-Bruhat et al., 1989). In simple terms, given a manifold $M$ (that could be Riemannian), a moving frame associates to each point in $M$ a reference frame (in general, non-inertial), and a Cartan’s connection is a derivative field of a moving frame that generalizes the concept of gradient. A common physical interpretation (in Mechanics) is that a Cartan’s connection generalizes the angular velocity of the moving frame (or of a moving rigid body) to a Riemannian manifold, and it contains geometric information, which is related to the curvature of the space (Sharpe, 1997). Cartan’s connection is a central concept in modern Particle Physics, as it can represent electromagnetic potential, gluons (particles that carry the strong force) and many others (Schwarz, 1996). In the case of Screw Theory, moving frames are fields of screws, and Cartan’s connections are fields of twists.

Connections are essential in the definition of time derivatives in non-inertial moving frames (covariant derivative or total derivative), and by using covariant derivative, it is possible to rewrite Newton’s law and Euler’s equation in a form that is invariant in any non-inertial frame. This framework even predict that serious troubles must be tackled in a possible generalization of Robotics to the Riemannian case, as it is easy to prove (and will be done in this work) that no reference frame exists that can be used as base of a kinematic chain (propagation of velocities and accelerations) like is done in the Euclidean space.

The objective of this work is manifold, but could be roughly divided in the following objectives: 1) convince the reader that angular velocities and twists are generalized by the concept of Cartan’s connection; 2) convince the reader that the Newton/Euler’s equations could be formulated by using the concept of covariant derivative; 3) To show that this way of formulating Screw Theory is the correct way to generalize for the case of Riemannian geometry, as well as show the problems that must be surpassed) and 4) Present some formulas (and its proofs) that this author
could not find in the specialized literature. Concepts and calculations will be presented to support items 1), 2) and 3), as well as references.

The point of departure of this work was the invariant Newton-Euler’s formulation of rigid body dynamics presented by (Sattinger and Weaver, 1986), section 14, pages 44-48, that is adapted and presented in section 2. It treats the kinematics and dynamics of a non-inertial frame (that could represent a rigid body) that only performs rotations around a fixed point. The precise notion of connections and covariant derivative is presented, as well as its dynamics. In section 3, it is presented the basic idea of the kinematic and dynamic in $SE(3)$, which describes the complete movement of points and rigid bodies in the Euclidean three-dimensional space $E^3$. The reader will easily recognize Screw Theory concepts, but it is a preparation for the next sections.

In section 4, it is presented important concepts of Riemannian spaces, that are useful to generalize Robotics to Riemannian space, and it is shown that the kinematics calculation are in trouble in this case, due to the space curvature (there is no inertial reference frame). In section 5, after the results obtained in the previous sections, it is shown how to determine kinematics and dynamics of manipulators (several frames) in Euclidean space (in the Cartan’s connections context), and some formulas are proved (that are believed not to be in the literature). Finally, in section 6, conclusions and directions of future works are presented.

2 Kinematics and Dynamics in $SO(3)$

Following the steps of the book (Sattinger and Weaver, 1986), any point in the Euclidean space $E^3$ is represented by a three dimensional vector $r'$ in a non-inertial frame $S'$, and by a vector $r$ in the inertial frame $S$. Both frames have the same origin. The transformation matrix between the two frames, that depends on time, is represented by $R = R(t)$, whose columns are the vectors of $S'$ expressed in the coordinates of $S$. $R(t)$ could be viewed as a trajectory in the Lie group $SO(3)$. The positions of an arbitrary point in $E^3$ are related by:

$$r(t) = R(t)r'(t)$$  \hspace{1cm} (1)

If one applies the time derivative in equation (1), one has

$$\dot{r} = \dot{R}r' + R\dot{r'} = R \left( R^T \dot{R}r' + \dot{r}' \right) = R \left( \Omega r' + \dot{r}' \right)$$  \hspace{1cm} (2)

where $\Omega = R^T \dot{R}$ is an anti-symmetric (time variant) matrix called body angular velocity, that is a trajectory in the Lie algebra $so(3)$, that could be called hodograph. Both time variant matrices $R(t)$ and $\Omega(t)$ must be seen as time variant operators in space $E^3$ that are representations of group $SO(3)$ and Lie algebra $so(3)$ in the vector space $E^3$. In fact, kinematics and dynamics of points use representations in $E^3$, but kinematics and dynamics of rigid bodies uses the adjoint representation, that are Lie groups and algebras representations in the very Lie algebra (in this case $so(3)$), which is also a vector space. Another important fact is the Lie algebra isomorphism between $so(3)$ and $\mathbb{R}^3$, with the Lie bracket $[A,B] = AB - BA$ corresponding to the vector product $\times$ of Analytic Geometry.

The action of $so(3)$ in $E^3$ given by applying $\Omega$ in $r'$, (that is $\Omega r'$), is equivalent to the vector product $\omega \times r'$, where there is a isomorphism between vector $\omega$ and matrix $\Omega$. The vector $\dot{r}'$ is called relative velocity, and equation (2) shows the transformation formula from the non-inertial frame to the inertial one for the velocities.

The matrix $\Omega(t)$ contains the information of how the (moving) frame $S'$ moves, and in the book (Sattinger and Weaver, 1986), it is said that it is a (Cartan’s) connection, although the point of view to be presented here are not developed there (and, as said before, the body is fixed by a point). Also, as the space is Euclidean, Cartan’s connection reduces to a Maurer-Cartan form restricted to a curve $R(t)$, that is a special kind of connection (from the didactic point of view, the understanding of Maurer-Cartan is a preparation for understanding connections).

In order to further discuss Maurer-Cartan (see, for example, (Ivey and Landsberg, 2003)), it is important to remember the following: given a Lie group $G$, its Lie algebra $\mathfrak{g}$ is the tangent space in the identity element $I \in G$, like presented in figure 1 – a). Any tangent vector in $G$, represented by a velocity $\Gamma$ of a curve $\Gamma(t)$, can be translated to the identity element $I$ by a left multiplication by $I$. This mapping is called left translation, and the formula $\Gamma = I \Gamma$ allows to represent any velocity $\Gamma$ as elements of the Lie algebra $\mathfrak{g}$. It is clear that it is the generalization, for an arbitrary group, of the formula $\hat{R} = R\Omega$ in $SO(3)$ (body’s velocity) presented in the last section (all groups share this property).

![Figure 1: Transformation Between Frames](image-url)

As $\Gamma(t)$ is invertible, the above formula can be written as $\Gamma^{-1} \dot{\Gamma} = \dot{\xi}$ and the (left) Maurer-Cartan form is defined by $\Xi = \Gamma^{-1} d\Gamma$, where $\Gamma$ is a generic matrix of the group (Sattinger and...
Weaver, 1986). It is clear that those formulas are related. Although it is tempting to divide both sides of the last formula by dt, it is not mathematically rigorous. In the following, it is illustrated the determination of a Maurer-Cartan form that is mathematically rigorous.

**Examples:** [Group SO(2)] This one-dimensional group is the planar rotation and corresponds to (or is a representation of) the unitary circle $S^1$, as depicted in figure 1–b). To any point in $S^1$ (except one) there is a function $\theta : S^1 \to \mathbb{R}$ that associates an number in the interval $[0, 2\pi)$ (that is, an angle) to that point. If $S^1$ is the unitary circle in the complex plane, any point in $S^1$ (except for one) is given by $R(\theta) = e^{i\theta}$, or:

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

(3)

In figure 1–b), it is represented a tangent (velocity) vector field $\dot{\theta}(\partial/\partial \theta)$, where $\partial/\partial \theta$ is the natural vector field such that $d\theta(\partial/\partial \theta) = 1$. A curve $R(t)$ in this group is equivalent to a curve of the form $R(t) = e^{i\theta(t)}$, where $\theta(t)$ is the angular time function. The Maurer-Cartan form in this case is deduced in the following way: given the matrix $R$ in equation (3), the differential is given by

$$dR = \begin{bmatrix} -\sin \theta & -\cos \theta \\ \cos \theta & -\sin \theta \end{bmatrix} d\theta$$

Then, the Maurer-Cartan form is given by

$$\Xi = R^{-1} dR = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

(4)

If one applies the Maurer-Cartan form to the vector field $\dot{\theta}(\partial/\partial \theta)$, one has

$$\begin{bmatrix} 0 & -\Delta \theta(\partial/\partial \theta) \\ \Delta \theta(\partial/\partial \theta) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}$$

which is the definition of $\Omega(t)$.

In a certain sense, one could say that the (angular) velocity field $\dot{\theta}(\partial/\partial \theta)$ is transformed in a hodograph by the Maurer-Cartan form (note that it is unidimensional). Returning to the general case of SO(3), it is possible to define:

**Definition 2.1 (Covariant Derivative)**

*Given a connection $\Omega$, that in the Euclidean case is a Maurer-Cartan form, the covariant derivative (or total derivative) is an operator that act in vectors and is given by:

$$D_t = \frac{\partial}{\partial t} + \Omega$$

In fact, if one applies $D_t$ to $r'$, it results in $\Omega r' + \dot{r}'$, which is the velocity of point $r'$ as seen in the inertial frame, but in the coordinates of the non-inertial frame $S'$.

Another time derivative applied in equation (1) would produce the transformation formula for accelerations, that involves several additional inertial accelerations, that corrects the Newton’s law for non-inertial systems. The result is very simple to obtain, but the same could also be obtained by:

**Definition 2.2 (Acceleration of Connection)**

*The acceleration for a given connection $\Omega$, that applies to a particle with mass $m$ is given by:

$$a' = D_t^2 r' = \left( \frac{\partial}{\partial t} + \Omega \right) (\Omega r' + \dot{r}') = \dot{r}' + \Omega \dot{r}' + 2\Omega r' + \Omega^2 r'$$

where $\dot{r}'$ is the relative acceleration, $\Omega^2 r'$ is the centrifugal acceleration, is $2\Omega \dot{r}'$ the Coriolis acceleration. This is the acceleration of point $r'$ as seen in the inertial frame $S$ but expressed in the non-inertial frame $S'$.

**2.1 Dynamics of points in SO(3)**

In order to describe the dynamic of a point with mass $m$, the Newton’s law is reformulated as

$$mD_t^2 r' = f'$$

(6)

where $f'$ represents the resulting force in the non-inertial frame $S'$. This could also be written in terms of the momentum of the particle, calculated as $P' = mD_t r'$, and the Newton’s law would be reformulated as $D_t P' = f'$. It is easy to see that all the inertial forces will appear.

**2.2 Dynamics of rigid body in SO(3)**

In order to obtain the dynamics of the rigid body motion, it is necessary to work in another representation of the Lie group SO(3) and Lie algebra $\mathfrak{so}(3)$, that is the adjoint representation in the very Lie algebra $\mathfrak{so}(3)$. In this new representation, the covariant derivative must be defined by:

**Definition 2.3 (Covariant Derivative)**

*Given a connection $\Omega$, the covariant derivative in the adjoint representation is an operator that act in elements of $\mathfrak{so}(3)$ given by:

$$D_t = \frac{\partial}{\partial t} + [\Omega, \cdot]$$

(7)

where $[A, B] = AB - BA$ is the commutator.

There are two copies of $\mathfrak{so}(3)$ that must be considered: 1) the copy where lives angular velocity and the quantities related, and 2) the copy where lives vector positions, point velocities and
etc. An essential concept in the dynamic formulation is the angular momentum, that is related to the angular velocity by the linear operator \( \Omega \), that is called the inertial operator. In the book (Sattinger and Weaver, 1986), it is shown how to calculate it, by working in the second copy of \( \mathfrak{s}\mathfrak{e}(3) \), that we avoid here. The angular momentum is given by

\[ \Lambda = \mathbb{I}(\Omega) \]

and the Newton’s law (or Euler’s) is given by:

\[ D_t \Lambda = T \]

where \( T \) is the torque matrix acting in the rigid body. After some substitution, we have

\[ D_t \Lambda = D_t \mathbb{I}(\Omega) = \frac{\partial}{\partial t} \mathbb{I}(\Omega) + [\mathbb{I}(\Omega), \mathbb{I}(\Omega)] = \mathbb{I}(\dot{\Omega}) + [\mathbb{I}(\Omega), \mathbb{I}(\Omega)] = T \]

After choosing a basis for \( \mathfrak{s}\mathfrak{e}(3) \) given by

\[ E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

and decomposing \( \Omega = \Omega_1 E_1 + \Omega_2 E_2 + \Omega_3 E_3 \), it is possible to show that \( \mathbb{I}(E_i) = I_i E_i \), where \( I_i \) are eigenvalues of the operator (until here, the Einstein convention is not used. Begin using here!).

As \( \Omega = \Omega E_i \), then

\[
\begin{align*}
\dot{\Omega}_i \mathbb{I}(E_i) & + [\Omega_j E_j, \Omega_k \mathbb{I}(E_k)] = \\
\dot{\Omega}_i I_i E_i & + [\Omega_j E_j, \Omega_k I_k E_k] = \\
(\dot{\Omega}_m I_m + I_k \Omega_k \epsilon_{ikm}) E_m & = T_m E_m
\end{align*}
\]

where \( \epsilon_{ikm} \) is the totally anti-symmetric tensor. This results in (the Euler’s equation):

\[
\begin{align*}
I_1 \dot{\Omega}_1 & = (I_2 - I_3) \Omega_2 \Omega_3 + T_1 \\
I_2 \dot{\Omega}_2 & = (I_3 - I_1) \Omega_3 \Omega_1 + T_2 \\
I_3 \dot{\Omega}_3 & = (I_1 - I_2) \Omega_1 \Omega_2 + T_3
\end{align*}
\]

In the next section, the calculations presented here (that came from (Sattinger and Weaver, 1986)) will be generalized to \( SE(3) \), that is, for screws, twists and wrenches. Although the formulas are well known in Screw Theory, the author could not find a similar formulation in the literature (using covariant derivative and connections).

3 Kinematics and Dynamics in \( SE(3) \)

3.1 Basics of \( SE(3) \) Kinematics

Again, we attach a reference frame \( S' \), that is non inertial, to a rigid body (that is, the body frame), and also a vector to describe the origin of the frame in relation to an inertial frame \( S \) (the space frame). Again, the three vectors of \( S' \) are arranged as column vectors in an orthogonal matrix \( R \), that belongs to the Lie group \( SO(3) \) (in planar movement, the group would be \( SO(2) \)). The origin of \( S' \) in relation to \( S \) is represented by a vector \( \gamma \in E^3 \), that are arranged in a larger \( 4 \times 4 \) matrix of the form

\[
\Gamma = \begin{bmatrix} R & \gamma \\ 0 & 1 \end{bmatrix}
\]

which belongs to the Lie group \( SE(3) \). Another point of view is important here, that was not used yet. To each \( \gamma \), that indicates a point in \( E^3 \), it is associated a copy of the Lie group \( SO(3) \) (let us call it \( SO(3)_\gamma \)). The Lie group \( SE(3) \) can then be identified with the space \( E^3 \times SO(3) \), which is of course, another Lie group (every vector space is a particular kind of Lie group, known as Abelian Lie group). \( SE(3) \sim E^3 \times SO(3) \) is also a special kind of differentiable manifold known as (principal) fiber bundle. A section of a (principal) fiber bundle can be imagined as a field that associates to each point in \( E^3 \) an element of \( SO(3) \), which is a frame. Any smooth field of frames represents possible rigid body motions in all directions. A curve in \( SE(3) \), that is \( \Gamma(t) \), describes the motion of a rigid body (translational and rotational, see, for example, (Ivey and Landsberg, 2003)).

As is done in Screw Theory (see for example (Selig, 2005)), the group action/representation of \( SE(3) \) in operators over \( E^3 \) represents, in the active point of view, a translation and rotation (screw movement) of a vector in \( E^3 \) and, in the passive point of view, a coordinate transformation from \( S' \) to \( S \). If Lagrangean dynamic formulation would be applied, \( SE(3) \) would be the configuration space for the rigid body motion. The space \( E^3 \) is homogeneous in the sense that no point is different of all the others (see (Kobayashi and Nomizu, 1969)). Every homogeneous space is the module of a Lie group and one of its sub-groups. In particular, \( E^3 = SE(3)/SO(3) \). The motion of points are simply curves in \( E^3 \).

If equation (1) is modified to include frame translation, then equations (2) could be written as \( \dot{r} = RD_t r' + \dot{\gamma} \). In the group \( SE(3) \), this expression is written as

\[
\begin{bmatrix} \dot{r} \\ 0 \end{bmatrix} = \begin{bmatrix} R & \gamma \\ 0 & 1 \end{bmatrix} D_t \begin{bmatrix} r' \\ 1 \end{bmatrix}
\]

where

\[
D_t = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial t} + \begin{bmatrix} \Omega & R' \dot{\gamma} \\ 0 & 0 \end{bmatrix}
\]

is the covariant derivative \( D_t \) to this new group and connection that will be analyzed in the sequel.
(in this case, the Cartan’s connection continues to be a Maurer-Cartan form, but for the group $SE(3)$).

**Theorem 1** Given two reference frames $S$ and $S'$ in an Euclidean space $E^3$, such that the first one is inertial, the frame curve $\Gamma(t)$ is a curve in $SE(3)$ and its velocity curve $\dot{\Gamma}$ has an hodograph in $\mathfrak{se}(3)$, which is also a curve of twists:

$$\xi(t) = \begin{bmatrix} \Omega & R^T \dot{\gamma} \\ 0 & 0 \end{bmatrix}$$

where $\Omega(t)$ is the rotation matrix curve in $\mathfrak{so}(3)$.

**Proof:** In order to deduce the formulas for transformation of velocities, the action of $SE(3)$ in the space $E^3$ rotates and translates the vectors:

$$\begin{bmatrix} r \\ 1 \end{bmatrix} = \begin{bmatrix} R & \gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r' \\ 1 \end{bmatrix}$$

that relates the coordinates of an arbitrary point in $S'$ and $S$. By differentiating both sides of (9), one has:

$$\begin{bmatrix} \dot{r} \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{R} & \dot{\gamma} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r' \\ 1 \end{bmatrix} + \begin{bmatrix} R & \gamma \end{bmatrix} \begin{bmatrix} \dot{r}' \\ 0 \end{bmatrix}$$

where the second formula at right is the action of $SE(3)$ in tangent spaces $T_vE^3$, and the first will be analyzed in que sequel. By applying the formula $\dot{R} = R\dot{\Omega}$, all the tangent vectors along the curve $\Gamma(t)$ can be transformed to elements in $\mathfrak{se}(3)$. This curve $\xi(t)$ is called **hodograph**. The Lie algebra $\mathfrak{se}(3)$ has a semi-direct decomposition $t(3) \ltimes \mathfrak{se}(3)$, where $t(3)$ is an abelian algebra (representing translational velocities) and the second, not abelian, representing rotations. It is also the Levi decomposition of $\mathfrak{se}(3)$ (Sattinger and Weaver, 1986). The formula in equation (10) can be simply written as:

$$\begin{bmatrix} \dot{r} \\ 0 \end{bmatrix} = \begin{bmatrix} R\Omega & \dot{\gamma} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r' \\ 1 \end{bmatrix} + \begin{bmatrix} \Omega & R^T \dot{\gamma} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{r}' \\ 0 \end{bmatrix}$$

That could even be written as:

$$\begin{bmatrix} \dot{r} \\ 0 \end{bmatrix} = \begin{bmatrix} R & \gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Omega & R^T \dot{\gamma} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r' \\ 1 \end{bmatrix} + \begin{bmatrix} \dot{r}' \\ 0 \end{bmatrix}$$

where the last term in the sum is the relative velocity, the matrix in (8) appears multiplying a position, which means that it has the role of an angular velocity.

Evidently, to describe acceleration of a point, as well as its dynamics, the equations (5), (6) and (7) should be reformulated to the new definition of connection, as presented in equation (8).

### 3.2 Dynamics of Rigid Bodies

A twist $\xi \in \mathfrak{se}(3)$ is related to a screw by the exponential formula $e^\xi$ (Murray et al., 1994). Chasles’ theorem states that any rigid movement is a screw. In a sufficiently small neighborhood of $I \in SE(3)$, there is a one-to-one correspondence between twists and screws, that is established by the exponential functions of Lie groups (see (Sattinger and Weaver, 1986)). If $g(t) \in SE(3)$ is the motion of a rigid body in relation to an inertial frame, $\Gamma^{-1}(t)\dot{\Gamma}(t) \in \mathfrak{se}(3)$ is called the body velocity and $\Gamma^{-1}(t)\dot{\Gamma}(t) \in \mathfrak{se}(3)$ the spatial velocity, and they are related by the adjoint action of $SE(3)$ in $\mathfrak{se}(3)$. The same ideas presented in subsection 2.2 apply in this case, which results in the dynamic equations for rigid bodies. As generalized forces are 1-forms in the configuration space (Bullo and Lewis, 2005), cotangent vectors in $SE(3)$ are generalized forces in a rigid body. Due to the Maurer-Cartan form (a particular type of connection), all the generalized forces can be mapped, by right and left translations, to the Lie co-algebra $\mathfrak{se}(3)^*$, and those elements are called **wrenches**. A wrench $\mathfrak{F}(t)$ acting (as a co-vector) on the rigid body with twist $\xi(t)$ (the vector) is the number $(\mathfrak{F}, \xi(t))$, which is the instantaneous work done by the wrench. In section 5, a continuation of this section will be presented.

### 4 Kinematics in Riemannian Space

In this section, the Cartan’s connection will be presented in its full generality (that is, it will not be a Maurer-Cartan form anymore) as the space where the bodies move, that is $M$, is Riemannian. Also, an arbitrary number of moving frames (as necessary in Robotics) will be assumed. In principle, none of them will be inertial, and it will be shown that it represents a big difficulty for the generalization of kinematic chains, that needs an inertial frame of reference.

In fact, $M$ is not a homogeneous space anymore, and cannot be given by module of two groups. The space of all frames in all points is no longer a group, but a principal fiber bundle $P$ with base $M$ and canonical projection $\pi: P \rightarrow M$, with fibers isomorphic to $SO(3)$, which is the symmetry group (Kobayashi and Nomizu, 1969), (Ivey and Landsberg, 2003). The action of $SO(3)$ in $P$ is **fiber preserving**, that is, only rotates a frame,
and not change its origin. The motion of a point is a curve \( r(t) \) in \( M \). A rigid body motion is a curve \( \Gamma(t) \in P \), and its projection \( \gamma(t) = \pi(\Gamma(t)) \) is the translational motion of the body. The principal fiber bundle has a symbolic notation, that is \((P, M, \pi, SO(3))\) (Kobayashi and Nomizu, 1969).

A moving frame \( P \) is a section \( \{ \text{field} \} \) that attributes to each point \( x \in M \) a reference frame \( s_x \). This could be seen as an application \( s : M \to P \) that satisfy \( \pi \circ s = \text{Id}_M \) (the identity application in \( M \)). A moving frame could also be seen as a set of tangent vector fields \( \{ x_i \} \), that are linearly independent in each point (the number of vectors is the dimension of \( M \)). Any other moving frame could then be written in terms of those vectors, which results in a field of \( SO(3) \) matrices \( X : M \to P \).

The concept of Cartan’s connection, in its full generality, appears in the following way: to a given moving frame \( s \), it is associated a 1-form field \( \Omega \) with values in \( \mathfrak{so}(3) \), that represents the frame angular velocity. It is formally defined as the Darboux’s derivative of the moving frame, and it is calculated as \( s^*(\psi) \) (the pullback of \( \psi \) by \( s \)), where \( \psi \) is the Maurer-Cartan form of \( SO(3) \) (Ivey and Landsberg, 2003).

Definition 4.1 (Cartan’s Connection) Given a Riemannian space \( M \), a Cartan’s connection \( \Xi \) is a field of 1-forms, with values in some Lie algebra \( \mathfrak{g} \) of some Lie group \( G \), which contains the geometric information of how frames change in any direction.

The Cartan’s connection is no longer a Maurer-Cartan form of a group (as was in the \( SE(3) \) case) but is related to such a form (in fact, it appears in connections’ definition). In fact, when a principal fiber bundle reduces to a Lie group, a Cartan’s connection reduces to a Maurer-Cartan form (Sharpe, 1997). A natural question to ask is: what are the conditions for a smooth field \( s : M \to P \) be a field of frames? The answer:

Theorem 2 (Cartan’s Theorem) Let \( G \) a matrix Lie group and its Lie algebra \( \mathfrak{g} \) with Maurer-Cartan form \( \psi_G \). Let also be a manifold \( M \) and a 1-form \( \Omega \) with values in \( \mathfrak{g} \) such as \( \text{d}\Omega + \Omega \wedge \Omega = 0 \) (\( \wedge \) is the exterior product). Then, in each point \( x \in M \), there is a neighborhood \( U \) and an application \( s : U \subset M \to G \) such that \( s^*(\psi_G) = \Omega \). Also, given two such applications \( s_1, s_2 \), they must satisfy \( s_1 = a s_2 \) for some \( a \in G \).

The equation \( \text{d}\Omega + \Omega \wedge \Omega = 0 \) is called the structural equation, and is satisfied by only in euclidian spaces (by Maurer-Cartan forms). Any other connection will not fit, which means that there exists a 2-form \( \Theta \) with values in \( \mathfrak{g} \) called curvature which is equal to \( \Theta = \text{d}\Omega + \Omega \wedge \Omega \) and measures how much the structural equation is not satisfied. For the rigid body motion, the angular velocity curve \( \Omega(t) \), that is calculated from the connection similarly as done in the example, is no longer a curve in \( \mathfrak{g} \) (in this case \( \mathfrak{so}(3) \)), but has its values distributed along copies of \( \mathfrak{so}(3) \) in each point of \( M \). Hodographs are not defined in this cases (Kobayashi and Nomizu, 1969), (Sharpe, 1997). Now, the transformation formulas between frames and Cartan’s connections are deduced (see figure 2).

**Theorem 3 (Frame Transformation)** Given two arbitrary reference (moving) frames \( S, S' \) (neither necessarily inertial), and \( X, X' \) its matrices of frames, \( X = XA \) the relation between them, and \( \nabla X' = X' \Xi, \nabla X = X \Xi \) its directional derivatives, the relation between the fields are given by:

\[
\Xi' = A^{-1} \text{d}A + A^{-1} \Xi A
\]

and is called a frame (gauge) transformation, and \( \Xi, \Xi' \) are two connections.

**Proof:** The formula in (13) can be easily demonstrated (Spivak, 2005):

\[
\nabla X' = X' \Xi' = \nabla (XA) = X \text{d}A + \nabla XA = X \text{d}A + X \Xi A
\]

which results in \( \Xi' = A^{-1} \text{d}A + A^{-1} \Xi A \).

Then, the formula in the theorem is a transformation between connections in different frames.

Definition 4.2 (Covariant Derivative) Given a (Riemannian) space \( M \) and a connection \( \Xi \) with values in \( \mathfrak{g} \), the associated covariant derivative is given by:

\[
D_t = \mathbf{i} \frac{\partial}{\partial t} + \Xi
\]

where \( \mathbf{i} \) is the identity in \( \mathfrak{g} \).

If three moving frames are involved (see figure 2), which is necessary in Robotics, the following results are important. Suppose there are three moving frames \( S, S' \) and \( S'' \), with matrices \( X, X' \) and \( X'' \), and the transformation matrices from one frame to another \( X = XA \) and \( X' = X' A' \). Consider also its directional derivatives given by \( \nabla X = \nabla X = X \Xi \) and \( \nabla X'' = X' \Xi' \), where the matrices \( \Xi, \Xi' \) and \( \Xi'' \) are the Cartan’s connection. Then:

**Proposition 4.1 (Transformation Formulas)** Given the moving frames \( S, S' \) and \( S'' \), then the transformation formula for the connections are:

\[
\Xi'' = (AA')^{-1} \text{d}(AA') + (AA')^{-1} \Xi (AA')
\]
5 Returning to Kinematics in \( SE(2) \)

In this final section, it will be shown, in the context of Screw Theory in Euclidean planar space, how the various formulas in kinematic and dynamic chains can be deduced in the context of Cartan’s connections (in this case, Maurer-Cartan forms). To the best of the author knowledge, the theorems and lemmas in this section, in the context of Cartan’s, are not in the literature.

In case of Euclidean space, one has:

\[
\Xi'' = (AA')^{-1} d(AA') + (AA')^{-1} 0(AA')
\]

(17)

In the particular case of frames with the same origin (that is, geometry of \( SO(3) \), it is easy to prove the following lemma.

**Lemma 5.1**

Given the matrix \((RR')^{-1} d(RR')\), it reduces to \((R')^T \Omega R' + \Omega'\), and \(\Omega + \Omega\) for the case of \( SO(2) \).

The covariant derivative in this case is obviously defined, and the following theorem shows that the Newton’s law can be correctly given by a generalization of equation (6).

**Theorem 4**
The Newton’s law in the frame \( S'' \) has the expression \( f'' = m(D_t)^2 r'' \) with covariant derivative given by:

\[
D_t = \frac{\partial}{\partial t} + (R')^T \Omega R' + \Omega'
\]

(18)

**Proof:** By applying two times the time derivative in the expression \( r = RR' r'' \), one has

\[
\hat{r}'' + 2 \hat{\Omega} r'' + 2(R')^T \Omega R' \hat{r}'' + 
\left[ (\Omega' + (R')^T \hat{\Omega} R' \right] r'' + 
\left[ (\Omega')^2 + (R')^T \Omega R' \right] r'' + 
2(R')^T \Omega R' \hat{\Omega}' r''
\]

Applying now the formula (18) two times, one has

\[
(D_t)^2 r'' = \left( \frac{\partial}{\partial t} + (R')^T \Omega R' + \Omega' \right)^2 r'' = 
\left( \frac{\partial}{\partial t} + (R')^T \Omega R' + \Omega' \right) \left( \hat{r}'' + (R')^T \Omega R' \hat{r}'' + \Omega' r'' \right)
\]

After some manipulation, it is easy to show that both formulas coincide, which concludes the proof.

**Corollary 5.1**

In case of \( SO(2) \) (planar movement), one has

\[
\hat{r}'' + 2 \left( \hat{\Omega}' + \Omega' \right) r'' + \left( \hat{\Omega}' + \hat{\Omega} \right) r'' + \left( \hat{\Omega}' + \Omega' \right)^2 r''
\]

**Proof:** By applying the commutative property of \( \Omega \) and \( \Omega' \) with other matrices, it is easy to complete the proof.

In case of \( SE(2) \) (that is, planar movement), one has:

**Theorem 5**
The Maurer-Cartan form for \( SE(2) \) in frame \( S'' \) has expression

\[
\xi'' = \left[ \begin{array}{ccc} \Omega + \Omega' & (R')^T (\Omega \gamma + R^T d\gamma + d\gamma) & 0 \\
0 & 0 & 0 \\
\end{array} \right]
\]

(19)

**Proof:** By calculating the inverses of a general matrix in \( SE(2) \) and its exterior derivatives \( dA, dA' \) and substituting these formulas in (19), one has:

\[
\xi'' = \left[ \begin{array}{ccc} (R')^T & -(R')^T \gamma' & R^T - R^T \gamma \\
0 & 1 & 0 \\
\end{array} \right] \left[ \begin{array}{ccc} dR' & d\gamma' & R' \gamma' \\
0 & 0 & 0 \\
1 & 0 & 0 \\
\end{array} \right]
\]

\[
+ \left[ \begin{array}{ccc} R & \gamma & dR \gamma \\
0 & 0 & 0 \\
\end{array} \right]
\]
\[
\zeta' = \begin{bmatrix}
(R')^T R T & -(R')^T R T^{'\gamma} - (R')^T \gamma' \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
(\text{d}R) R' + R \text{d}R' & -dR \gamma' + d\gamma + R \text{d}\gamma' \\
0 & 0
\end{bmatrix}
\]

By evoking lemma 5.1, the result follows.

6 Conclusions

It was presented some concepts and results relating the Screw Theory (used in Robotics) to concepts in Differential Geometry and Lie groups and Algebras. In particular, it was shown that Cartan’s connections (a differential geometric concept) is the natural generalization for angular velocities and twists for general Riemannian spaces. It was also shown that Newton-Euler formulation of Robotics are written in this context in an invariant form (by using connections) and more research has to be done in order to generalize Robotics to the Riemannian case (necessity to redefine kinematic and dynamic chains, as no inertial reference frame is possible in general). To the best of the author’s knowledge, such an exposition is lacking in the Robot’s Literature, as well as some of the formulas and theorem’s proofs.

In future works, the question of how to calculate angular velocities of robotic chains in Riemannian spaces, as well as questions related to the dynamics of rigid bodies, will be investigated, as well as possible practical applications of the results presented. Also, several other formulas and theorems had to be excluded from this paper, due to the lack of space in a conference paper, as well as important practical matters like singular configurations. In a (near) future extended version of this work, to be submitted to a journal, all those questions will have their deserved places.

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