TRAJECTORY TRACKING CONTROL OF NONHOLONOMIC WHEELED MOBILE ROBOTS WITH SLIPPING ON CURVILINEAR COORDINATES: A SINGULAR PERTURBATION APPROACH

C. A. Peña Fernández,* Jés J. F. Cerqueira* Antonio M. N. Lima†

*Robotics Laboratory - Department of Electrical Engineering, Polytechnic School, Federal University of Bahia
Rua Arístides Novis, 02, Feira, 40210-630, Salvador, Bahia, Brasil
Telefone:+55-71-3203-9760.

†Department of Electrical Engineering at Center of Electrical and Computer Engineering, Federal University of Campina Grande
Rua Aprigio Veloso, 882, Universitário, 58429-970, Campina Grande, Paraíba, Brasil
Telefone:+55-83-2101-1000.

Email: cesar.pena@ufba.br, jes@ufba.br, amnlima@dee.ufcg.edu.br

Abstract—This paper considers the trajectory tracking control of a wheeled mobile robot (WMR) with slipping in the wheels, i.e., when the kinematic constraints are not satisfied. The proposed controller guarantees that the tracking error converges to small ball of the origin such that the radius of this ball can be adjusted by selecting appropriate parameters. To this end, the controller is designed in two parts: the kinematic controller based in curvilinear coordinates and the dynamic controller based in a nonlinear state feedback. The singular perturbations approach allows to manipulate the flexibility through of a small factor in the dynamic model (normally, known as ε) at the same time that scales the dissatisfaction of the kinematics constraints. Thus, we will observe the behavior of the tracking resultant when the controller is applied to such model.

Keywords—Nonholonomic wheeled mobile robot, slipping, curvilinear coordinates, trajectory tracking control, singular perturbations.

Resumo—Este artigo aborda o problema de controle de seguimento de trajetória de um robô móvel (RMR) com deslizamento nas rodas, ou seja, com insatisfação das restrições cinemáticas. O controlador proposto garante que o erro de seguimento convirja a uma vizinhança da origem representada por uma bola cujo raio pode ser ajustado pela escolha de parâmetros apropriados. Para tal fim, o controlador é projetado em duas partes: o controlador cinemático baseado em coordenadas curvilineas e o controlador dinâmico baseado em uma realimentação não-linear de estados. O modelo dinâmico do RMR considerado neste artigo é formalizado pelo uso da teoria de perturbações singulares. A teoria de perturbações singulares não só permite manipular a flexibilidade dentro do modelo dinâmico através de um pequeno fator (usualmente conhecido como ε), mas também pondera a insatisfação das restrições cinemáticas. Dessa forma, neste artigo será observado o comportamento do controlador de seguimento de trajetória proposto quando este seja aplicado no modelo dinâmico.

Palavras-chave—Robô móvel não-holonômico, deslizamento, coordenadas curvilíneas, controle de seguimento de trajetória, perturbações singulares.

1 Introduction

In recent years, there has been a growing interest in the design of feedback-control laws for mechanical systems subjected to nonholonomic constraints. This is the case of the stabilization and tracking problems of wheeled mobile robots (WMRs). The stabilization has been an extensive research area in past decades due to its challenging theoretical nature, i.e., an intrinsic nonlinear control problem, and its practical importance. It is well-known that there does not exist a smooth pure state feedback control law such that the state of a wheeled mobile robot converges to the origin (Dong and Kuhnert, 2005; Fernández et al., 2013). In order to mitigate this difficulty, several types of controllers have been proposed, such as time-varying control laws, discontinuous control laws, and hybrid control laws (For more details, see (Bloch et al., 2000; Kolmanovskiy and McClamroch, 1995)).

The tracking problem of WMRs has also been studied. The techniques for trajectory control has been based in linearization techniques for local controlling (Walsh et al., 1994); in techniques of nonlinear state feedback with singular parameters (D’Andrea-Novels et al., 1995; Leroquais and D’Andrea-Novels, 1996; Motte and Campbell, 2000); or also in techniques based in backstepping (Jiang, 2000; Jiang and Nijmeijer, 1999).

In this paper, we consider the tracking control problem of WMRs which are subjected to slipping effects, i.e., when the nonholonomic kinematic constraint of pure rolling is transgressed during the motion. In principle, this is due to various effects such as deformability or flexibility of the wheels (Leroquais and D’Andrea-Novels, 1996; Fernández et al., 2012; Fernández and Cer-
queira, 2009b; Fernández and Cerqueira, 2009a). By considering these effects, we will study the trajectory tracking control of the WMRs with slipping in the dynamic and aim at designing a robust controller based on a nonlinear state feedback. To this end, the kinematics of the WMR is derived by considering the slipping and small deformations of the wheels. Such consideration allows to use the singular perturbations theory due to its powerful utility to insert small parameters which can be used to represent the flexibility (or wheel’s deformation) (D’Andréa-Novel et al., 1995). Thus, the dynamic of the WMR is given in formalism with the aid of Lagrange approach, like in (Fernández et al., 2013). Complementary, the singular perturbations theory is used to add a small scale factor that represents the flexibility (or deformation). However, some assumptions are made about the kinematic constraints and the scale factor so that the kinematic controller and the dynamic controller can be designed.

This paper is organized as follow: In Section 2 is showed the mathematical model and the preliminaries foundations associated with the singular perturbation theory. In Section 3 is presented the project of the controller divided in two parts, the kinematic controller based in curvilinear coordinates and the dynamic controller based in a nonlinear state feedback. In order to verify effectiveness of the proposed controller, in Section 4 a simulation is done. Finally, conclusions and final remarks are made in Section 5.

2 Dynamic model and theoretical preliminaries

This paper consider the configuration of a WMR with two controllable wheels (differential traction), as shown in Figure 1. Such configuration can be fully described by the vector $q \in \mathbb{R}^5$ of generalized coordinates defined by

$$q = \begin{bmatrix} x_1 & x_2 & \theta & \varphi_1 & \varphi_2 \end{bmatrix}^T$$

where $\{x_1, x_2, \theta\}$ is the set of coordinates associated with the cartesian position of the body frame $\{L\}$ into the global frame $\{G\}$ and guidance of mobile base, the set $\{\varphi_1, \varphi_2\}$ is associated with the angular position of each wheel (which can not be controlled independently) (see Fig. 1).

The kinematic constraints can be expressed like a Pfaffian constraint (Motte and Campion, 2000; Bloch et al., 2000):

$$A^T(q)\dot{q} = 0$$

where $A(q)$ is the matrix with the nonholonomic kinematic constraints, and defined by

$$A(q) = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix} \triangleq \begin{bmatrix} A_1(\theta) \\ A_2(\theta) \end{bmatrix} \triangleq A(\theta).$$

being $\mu = [\mu_1 \mu_2 \mu_3]^T$ an instrumental vector in sense of accessing the violations of the ideal kinematic constraints in the WMR (Fernández et al., 2013), $v = [v_n \omega]^T$ is the vector that contains the linear ($v_n$) and angular ($\omega$) velocities and

$$S(q) = \begin{bmatrix}
-\sin \theta & 0 & 0 \\
\cos \theta & 0 & 1 \\
-1/r & -b/r & 1/r & -b/r
\end{bmatrix} \triangleq \begin{bmatrix} S_1(\theta) \\ S_2(\theta) \end{bmatrix} \triangleq S(\theta),$$

is the Jacobian. The term $\varepsilon$ is a scale factor associated with the flexibility of the dynamic model (Fernández et al., 2013).

Property 1 The matrices $A(q)$ and $S(q)$ satisfy $A^T(q)S(q) = 0$ (D’Andréa-Novel et al., 1995; Murray et al., 1994).

As usual, the dynamic model for a WMR is given by

$$M \ddot{q} = \Lambda + Bu + A(\theta)\lambda$$

where

$$M = \text{diag}(m I_c I_w I_w), \quad \Lambda = 0_{5 \times 1}, \quad B = \begin{bmatrix} 0_{3 \times 2} \\ 0_{3 \times 2} \end{bmatrix}$$

are the inertia matrix, the centripetal and coriolis torques (It is assumed that the geometrical center coincides with the mass center, thus this vector is equal to zero) and a full rank matrix, respectively. The parameter $m$ is the mass of the WMR, $I_c$ is the moment of inertia of the WMR on a vertical axis through the intersection of the axis of symmetry with the driving wheel axis and $I_w$ is the moment of inertia of each driving wheel on its axis. The vector $u$ represents the input torques provided by the actuators and $\lambda \in \mathbb{R}^3$ represents the Lagrange multipliers (Bloch et al., 2000).
\[ A^T(q)\dot{q} = A^T(q)A(q)\varepsilon \mu. \] (4)

**Assumption 1** Assume that the norm of \( A^T(q)A(q)\varepsilon \mu \) is limited, i.e., \( \| A^T(q)A(q)\varepsilon \mu \| \leq \xi \), where \( \xi \) is a non-negative known function which depends on the lateral acceleration of the robot and the deformation of the wheels.

If \( \varepsilon = 0 \) then (4) becomes the ideal constraint (1). In other words, the parameter \( \varepsilon \) governs the dissatisfaction of the kinematic constraints and it must be included into the dynamic model. To this end, we propose a singularly perturbed dynamic model for the WMR, like in (Fernández et al., 2013), defined by the following state-space:

\[ \begin{cases} 
\dot{x} = B_0(q)v + \varepsilon B_1(q) + B_2(q)\mu + B_3(q)u_e \quad (5) \\
\varepsilon \mu = C_0(q)v + \varepsilon C_1(q) + C_2(q)\mu + C_3(q)u_e \quad (6) \\
y = P_0(q) \quad (7)
\end{cases} \]

where \( x = [q^T \; \dot{q}^T]^T \) can be used to denote the “slow” variables and \( \mu \) beyond its instrumental meaning can be used to denote the “fast” variables; \( u_e = [u_{e1} \; u_{e2} \; u_{e3}]^T \) has the manipulated inputs associated with the torques at the motors and \( y = [y_1 \; y_2]^T \) has the cartesian coordinates of a point \( p \) located at a distance \( L \) of the symmetry axis of the WMR, i.e., we define:

\[ y = [y_1 \; y_2] = P_0(q) \triangleq \left[ \frac{x_1 - L\sin \theta}{x_2 + L\cos \theta} \right]. \] (8)

The matrices \( B_i(q), C_i(q), i = 0, 1, 2, 3, \) are successively:

\[ B_0(q) = \begin{bmatrix} \delta(\theta) \\ \Delta_0 \end{bmatrix}, \quad B_1(q) = \begin{bmatrix} A(\theta) \\ \Delta_1 \end{bmatrix}, \]

\[ B_2(q) = \begin{bmatrix} 0 \sin \Delta_2 \\ \Delta_2 \end{bmatrix}, \quad B_3(q) = \begin{bmatrix} 0 \cos \theta \\ \Delta_3 \end{bmatrix}, \]

\[ C_0(q) = \begin{bmatrix} -\theta \cos \Delta_0 \\ 1/3 \theta \sin \Delta_0 \\ -1/3 \theta \sin \Delta_0 \end{bmatrix}, \quad C_1(q) = \begin{bmatrix} 0 & 0 & -\theta \\ 1/3 \theta & 0 & 0 \\ 1/3 \theta & 0 & 0 \end{bmatrix}, \]

\[ C_2(q) = \begin{bmatrix} a_1 D_0 & 0 & 0 \\ a_2 G_1 & a_2 G_2 & 0 \\ a_3 G_1 & a_3 G_2 & a_3 G_3 \end{bmatrix}, \quad C_3(q) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]

being

\[ \Delta_0 = \begin{bmatrix} 1/3 \theta \sin \Delta_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta_1 = \begin{bmatrix} -1/3 \theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]

\[ \Delta_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_2 G_1 & a_2 G_2 \end{bmatrix}, \quad \Delta_3 = \begin{bmatrix} -a_1 & a_1 & 0 \\ -a_1 & -a_1 & 0 \end{bmatrix}, \]

with

\[ a_1 = \frac{r^2}{3 I_w}, \quad a_2 = \frac{2 I_w b^2 - 2 I_w r^2}{I_w (\delta + V)} \]

\[ a_3 = \frac{4}{m (\delta + V)}, \quad a_4 = -\frac{2 b^2}{I_w (\delta + V)} - \frac{r^2}{I_w (\delta + V)} \]

where the parameter \( V \) is the velocity of the wheel center and \( \delta \) is a “small” positive constant to avoid the numerical problem for small values of \( V \) (i.e., for small values of \( V \), it is replaced by \( V + \delta \)). The parameters \( D_0 \) and \( G_0 \) are normalized values defined by

\[ D_0 = \varepsilon D \quad \text{and} \quad G_0 = \varepsilon G, \] (9)

where \( D \) and \( G \) are the stiffness coefficients for the transversal and longitudinal movements of each wheel, respectively.

**Assumption 2** The longitudinal and transversal stiffness coefficients \((G \text{ and } D, \text{ respectively})\) are the same for the three wheels and

\[ \varepsilon = \inf \{1/G, 1/D\} \].

**Assumption 3** The velocities of both driving wheels at their center are taken to be identical, and more precisely, equal to their average:

\[ V = \left( \frac{x_1^2 + x_2^2 + \theta^2}{3} \right)^{1/2}. \] (10)

**Remark 1** When \( \varepsilon = 0 \) the model defined by (5)-(7) is called rigid model. When \( \varepsilon \neq 0 \) the model is called flexible model (D’Andréa-Novot et al., 1995).

### 2.2 Problem of Trajectory tracking control in curvilinear coordinates

Given a differentiable simple curve \( C \) defined by one of its point, the unitary tangent vector at the point, and its curvature \( \text{curv}(s) \) where \( s \) is the curvilinear coordinate along the curve, the following assumptions will be considered in order to make the controller design easy (Dong and Kulharn, 2005).

**Assumption 4** Let \( |\text{curv}(s)| < 1/R, \forall s \) where \( R > 0 \) is a constant.

**Assumption 5** For a given point \( Q \) in the curve \( C \), assume the curvilinear coordinate at \( Q \) is \( s \), and \( \{Q, T(s), N(s)\} \) is the Frenet frame on the curve at point \( Q \), being \( T(s) \) the tangent vector at point \( Q \) and \( N(s) \) the normal vector at same point (see Fig. 2).

**Assumption 6** The distance between point \( p \) and the curve \( C \) is smaller than \( R \), thus the projection of point \( p \) on the curve is unique and denoted as \( Q \).

Let \( |pQ| \) be the distance between the two points \( p \) and \( Q \), \( \alpha \) be the orientation of the WMR with respect to the tangent vector \( T(s) \) of the curve \( C \) at point \( Q \), given a desired velocity \( v_n^d > 0 \) (see Fig. 2).
The control problem considered in this paper is finding a controller $u_c$ for system (5)-(7) such that $|pQ|, |α|$ and $|v_n - v_n^∗|$ are as small as possible when time approaches to the infinity.

The position of point $p$ is parameterized by $(s, d)$, where $d$ is the coordinate of point $p$ along $N(s)$. Noting $2^{\gamma} = \theta - \gamma$, the WMR’s configuration is parameterized by

$$q^* = [q_1, q_2, q_3]^T = [s, d, \alpha]^T. \quad (11)$$

By classic mechanics and also proposed in (Dong and Kuhnert, 2005):

$$\begin{align*}
\dot{q}_1 &= \frac{v_n \cos q_3 + \varepsilon \sin q_3}{1 - \text{curv}(q_1)q_2} \quad (12) \\
\dot{q}_2 &= v_n \sin q_3 - \varepsilon \cos q_3 \quad (13) \\
\dot{q}_3 &= \omega - \frac{v_n \text{curv}(q_1) \cos q_3}{1 - \text{curv}(q_1)q_2} - \varepsilon \text{curv}(q_1) \sin q_3 \\
\dot{q}_3 &= \frac{\varepsilon \text{curv}(q_1) \sin q_3}{1 - \text{curv}(q_1)q_2}. \quad (14)
\end{align*}$$

Noting the Assumption 4, the equations (12) and (14) are well-defined if $|q_2| < R$. In the controller design, this condition will be guaranteed.

## 3 Controller design

The controller is designed in two parts. The first part, a kinematic controller for subsystem defined by (12)-(14) is designed with the aid of an appropriate transformation. In the second part, a robust nonlinear state feedback based controller is proposed with the aid of the inverse dynamics and the controller obtained in the first part.

### 3.1 Kinematic controller

Let $e = [e_1, e_2, e_3]^T$ be the error of the tracking trajectory associated with $q^*$, formally defined by the following transformation

$$\begin{align*}
e &= [e_1, e_2, e_3]^T = \Pi(q) \quad (15) \\
w &= [w_1, w_2]^T = \Pi^{-1}(q)v \quad (16)
\end{align*}$$

Let

$$\begin{align*}
e_1 &= q_1 \\
e_2 &= \frac{2R}{\pi} \tan \frac{\pi q_2}{2R} \\
e_3 &= \mathcal{L}_{q_1}e_2 + \frac{\pi}{v_n} \quad (17)
\end{align*}$$

$$\begin{align*}
w_1 &= \frac{v_n \cos q_3}{1 - \text{curv}(q_1)q_2} \\
w_2 &= w_1 \left( \mathcal{L}_{q_1}e_2 + \frac{\mathcal{L}_{q_1}v_1 - k_2v_1}{v_n^*} + k_2v_3 \right. \\
&\left. - k_2^2e_2 + \omega \mathcal{L}_{q_1}e_2, \right.
\end{align*}$$

where $g_2 = [0, 0, 1]^T$, $\phi_1 = \xi \frac{\partial e_2}{\partial q_2}$, $\phi_2 = \xi \frac{\partial e_2}{\partial q_1}$, $\phi_3 = \xi \frac{\partial e_3}{\partial q_1}$, $\phi_4 = \xi \frac{\partial e_3}{\partial q_2}$, $\theta = \frac{\sin q_2}{1 - \text{curv}(q_1)q_2}$, $\sin q_3 = \frac{\cos q_3 - \text{curv}(q_1)\sin q_3}{1 - \text{curv}(q_1)q_2}$, $\cos q_3 = \frac{1 - \text{curv}(q_1)\sin q_3}{1 - \text{curv}(q_1)q_2}$, $\mathcal{L}$ is the abbreviation of Lie Derivative, i.e.,

$$\begin{align*}
\mathcal{L}_{q_1}e_2 &= \frac{\partial e_2}{\partial q_1} \quad (18) \\
\mathcal{L}_{q_1}e_2 &= \frac{\partial e_2}{\partial q_1} \quad (19)
\end{align*}$$

Assuming that $w_1$ and $w_2$ are control inputs, one has the following lemma.

**Lemma 1 ((Dong, 2010))** Assume $v_n^* > \delta_v > 0$, if

$$w = \eta, \quad (20)$$

then $e_2$ and $e_3$ converge exponentially to a small ball containing the origin. The radius of the ball can be adjusted by $\delta_1 > 0$, where $\eta = [\eta_1, \eta_2]^T$ and

$$\begin{align*}
\eta_1 &= v_n^*, \\
\eta_2 &= -\xi \phi_2 \tan \left( \frac{e_3 \xi \phi_3}{\delta_1} \right) - k_3 e_3 v_n^* - e_2 v_n^* + \frac{\varepsilon \phi_1}{(v_n^*)^2}.
\end{align*}$$

**Proof:** Let the Lyapunov function

$$V = \frac{1}{2} (e_2^2 + e_3^2)$$

differentiating it along the close-loop represented by (17)-(19), one obtains

$$\begin{align*}
\dot{V} &= -k_2 v_n^* e_2^2 - k_3 v_n^* e_3^2 - e_2 \phi_1 + e_2 \mathcal{L}_{q_1}e_2 \\
&\quad - e_3 \xi \tan \left( \frac{e_3 \xi \phi_3}{\delta_1} \right) + e_3 \varepsilon \phi_2.
\end{align*}$$
In the above expression the following inequalities are satisfied:

$$|\xi e_2 L_{g^2, e^2}| \leq \rho \delta_1$$

$$-e_3 \phi_2 \tanh \left( \frac{\xi e_3 \phi_2}{\delta_1} \right) + |e_3 \phi_2| \leq \rho \delta_1,$$

where $\rho$ is a constant which satisfies $\rho = e^{-(\nu+1)}$ (i.e., $\rho = 0.2785$). Thus,

$$\dot{V} \leq -k_2 u_\star^2 e_2^2 - k_1 u_\star^1 e_3^2 + 2\rho \delta_1$$

$$\leq -2\min\left( k_2 \delta_1, k_3 \delta_1 \right) V + 2\rho \delta_1.$$

Noting $u_\star^1 \geq \delta_1 > 0$ it can be noted that $V$ exponentially converges to a small ball containing the origin. The convergence rate is at least

$$\lim_{t \to \infty} \frac{\rho \delta_1}{2k_2 \delta_1, 2k_3 \delta_1}.$$

The radius of the ball can also be adjusted by the parameter $\delta_1$. Therefore, $e_2$ and $e_3$ converge exponentially to the small ball containing the origin, and the radius of the ball can also be adjusted by the control parameter $\delta_1$.

From (16) and (20) is obtained that

$$v = \Pi_2(q) w = \Pi_2(q) \eta.$$  \hspace{1cm} (21)

The equation (21) is so-called the kinematic controller for the WMR.

3.2 Dynamic controller

Like in (D’Andrée-Novel et al., 1995), assuming that the control input $u_c$ is a smooth function of time $u_c(t) = u_c(q, v)$ then, for $\varepsilon = 0$, the equation (6) can be rewritten as follows:

$$C_0(q)v + C_2(q)\mu + C_3(q)u_c(q, v) = 0,$$  \hspace{1cm} (22)

**Definition 1 (D’Andrée-Novel et al., 1995))**

The model defined by (5)-(7) is in standard form if only if (22) has $k \geq 1$ distinct isolated roots.

Indeed, the root of (22), here denoted by $\bar{\mu}$, is

$$\bar{\mu} = -C_2^{-1}(q) [C_3(q) u_c(q, v) + C_0(q)v],$$  \hspace{1cm} (23)

thus the reduced system associated is obtained by substituting (23) in (5):

$$\dot{x} = B_0(q)v - \left[ \varepsilon B_1(q) + B_2(q) \right] C_2^{-1}(q) [C_3(q) u_c(q, v) + C_0(q)v] + B_3(q)u_c;$$

$$\dot{x}(0) = x_0,$$  \hspace{1cm} (24)

and the boundary layer system is

$$\frac{d\bar{\mu}}{dt} = C_0(q)v_0 + [\varepsilon C_1(q_0) + C_2(q_0)] (\bar{\mu} + \bar{\mu})$$

$$+ C_3(q_0) u_c;$$

$$\bar{\mu}(0) = \mu_0 - \bar{\mu},$$  \hspace{1cm} (25)

where $t = t/\varepsilon, v_0, q_0$ are interpreted as fixed parameters and $\bar{\mu} = \mu - \bar{\mu}$ being $\mu_0$ equal to (23) evaluated in $v_0, q_0$.

Now, we introduce two conditions:

**Condition 1** There exist $T, \lambda_1, \lambda_2, \varepsilon_0$ and the balls $Z_1 = (0; \lambda_1)$, $Z_2 = (0; \lambda_2)$ such that

- The matrices $B_i(q)$ and $C_i(q)$ in the model (5)-(7) (for $i = 0, \ldots, 3$) and their partial derivatives with respect to $x, \mu$ and $\varepsilon$ are continuous in $Z_1 \times Z_2 \times [0, \varepsilon_0] \times [0, T]$,

- The function (23) and $\varepsilon C_1(q) + C_2(q)$ have continuous first partial derivatives,

- The reduced system (24) has an unique solution $\bar{x}$ defined on $[0, T]$ which belongs to $Z_1$.

**Condition 2** $\bar{\mu} = 0$ is an exponentially stable equilibrium point of the boundary layer system (25) uniformly in the parameter $x_0$. Furthermore, $\mu_0 - \bar{\mu}(0)$ belongs to its domain of attraction. This condition implies that $\bar{\mu}(\tau)$ exists for $\tau \geq 0$ and that

$$\lim_{\tau \to +\infty} \bar{\mu}(\tau) = 0.$$  \hspace{1cm} (26)

The Tikhonov’s theorem states the relation between $x$ and $\bar{x}$ on one hand and $\mu, \bar{\mu}$ and $\bar{\mu}$ on the other hand.

**Theorem 1 (Tikhonov’s theorem)** For a system in a standard form, if the Conditions 1 and 2 are satisfied, then there exist positive constants $\nu_1, \nu_2$ and $\varepsilon^*$ such that if $\|x_0\| \leq \nu_1, \|\mu_0 - \bar{\mu}_0\| \leq \nu_2$ and $\varepsilon < \varepsilon^*$ then the following approximations are valid for $t \in [0, T]$:

$$x(t) = \bar{x}(t) + O(\varepsilon)$$  \hspace{1cm} (27)

where $O(\varepsilon)$ represents a quantity of the order of $\varepsilon$.

The Theorem 1 implies that there exists $t_1 > 0$ such that the approximation

$$\mu(t) = \bar{\mu}(t) + O(\varepsilon)$$

is valid for $t \in [t_1, T]$. Leaving only choose an appropriate value for $\varepsilon$, such that the Theorem 1 is satisfied.

3.3 Computing the control law $u_c$

The global feedback control $u_c = u_c(q, v)$ is projected by using the inverse dynamics of (3) and the second derivative of (2). Thus,

$$\dot{\bar{q}} = \left[ \frac{\partial S}{\partial q} S(q)v \right] v + S(q) \ddot{v}.\hspace{1cm} (28)$$

Eliminating Lagrange multipliers in (3) and using the relation (28) give

$$\dot{v} = [S^T(q) MS(q)]^{-1} S^T(q) \left[ B u_c - M \left[ \frac{\partial S}{\partial q} S(q)v \right] v \right].\hspace{1cm} (29)$$
Consequently, the law \( u_e \) is defined by the inverse of (29):

\[
u_e = [S^T(q)B(q)]^{-1} \left\{ S^T(q) \left[ M(q)S(q)\rho + M(q)\frac{\partial S}{\partial q} \right] v \right\}.
\]

Remark 2 By substituting (30) in (29) is obtained

\[
\rho = \dot{v} - \frac{\partial \Pi(q)}{\partial q} \eta + \Pi(q) \ddot{\eta}.
\]

The proposed controller in (30) and (32) is a combination of the existing results in singular techniques when the slipping is included into the dynamic model (D’Andréa-Novel et al., 1995; Lerouquais and D’Andrea-Novel, 1996). Unlike contributions using manifolds of \( \mu \) in order to linearize the dynamic model (Motte and Campion, 2000), in this paper is included the flexibility (represented by parameter \( \varepsilon \)) within the kinematic model (see the inclusion of \( \varepsilon \) and \( \xi \) in (12)-(14) and (20), respectively).

4 Evaluating the controller

In the kinematic controller, the transformation \( \epsilon_2 = \frac{2R}{\pi} \tan \frac{\pi \rho}{2R} \) guarantees that \( |q_2| < R \) if \( \epsilon_2 \) is bounded. In other hand, if the one wants to make \( |q_2| < d_0 < R \) (where \( d_0 \) is a positive constant that represents the initial displacement of the point \( p \)) during the control, then one can set \( \epsilon_2 = \frac{2d_0}{\pi} \tan \frac{2\rho}{2R} \). It should be noted that the initial value of \( q_2 \) should be less than \( d_0 \) too. Otherwise, an open-loop control can be first applied to the system such that \( q_2 < d_0 \).

In order to verify effectiveness of the proposed controller, simulations were done by using the kinematic control defined by (32) and the robust nonlinear state feedback based controller defined by (30). However, we must first know the appropriate value for \( \varepsilon \) due that Theorem 1 imposes the limit \( \varepsilon^* \).

Remark 3 Phenomenologically, the value of \( \varepsilon \) is associated with the flexibility and deformation of the wheels, we can say that when the value of \( \varepsilon \) increases then the computational cost of the control law (30) also increases. This is a direct consequence of the friction coefficient, which also increases when the deformation of the wheels increases and it is a significant cause of the dead zone in the actuator. So, the computational effort is associated with an attempt of the control law (30) to overcome the dead zone (Fernández and Cerequeira, 2009b).

4.1 Choosing the value of \( \varepsilon^* \)

Let considered the following transformation on \( \varepsilon \) for a better numerical manipulation:

\[
\varepsilon = 10^{-n_\varepsilon} + N_\varepsilon 10^{-n_\varepsilon+1}
\]

being \( n_\varepsilon \in \mathbb{Z}_+ \) and \( N_\varepsilon \in [0,1] \subset \mathbb{R}_+ \). Assuming that the coefficients \( D \) and \( G \) are the same for all wheels and by using of (9) and the Assumption 2, we chosen \( D_0 = G_0 = 1 \ N \).

We assume that the desired trajectory is a rhombus with the diagonals equal to 6.28 m. Each simulation represent a duration of 4.5 s. The numerical values used in the simulations are the same so (Lerouquis and D’Andrea-Novel, 1996): \( m = 1000 \ Kg \), \( I_c = 500 \ Kg-m^2 \), \( I_w = 1.6 \ Kg-m^2 \), \( L = 1 \ m \), \( b = 1 \ m \) and \( r = 0.35 \ m \). In the kinematic controller we choose \( k_1 = k_2 = 1 \), \( \delta_1 = 0.01 \) and \( R = 10^3 \) m due that the rhombus has four corners (i.e., curv(\( s \)) \to \infty), thus \( R = 10^3 \) m simulates a quasi-infinite curvature.

In Fig. 3(b) is shown the computational cost (measured in seconds) for the interval \([10^{-16}, 9 \times 10^{-11}] \) in the domain of \( \varepsilon \). It can be seen that the evolution of the computational cost increases when \( \varepsilon \) increases. Particularly, when \( \varepsilon = 9 \times 10^{-11} \) (\( N_\varepsilon = 0.8 \) and \( n_\varepsilon = 11 \)) the system is unstable for the tracking, as shown in Fig. 3(a). Thus, it is defined \( \varepsilon^* = 9 \times 10^{-11} \) in the Theorem 1. Values greater than \( \varepsilon^* \) unstimulize the system.
4.2 Trajectory tracking

To observe the behavior of the control law (30) when it is applied in the model defined by (5)-(7), we can study the cases in which the model is totally rigid ($\epsilon = 0$) and flexible ($\epsilon \neq 0$), according to the Remark 1.

In Fig. 4(a) is shown the tracking made by the control law (30) when $\epsilon = 4 \times 10^{-11}$ such that the condition $\epsilon < \epsilon^* = (9 \times 10^{-11})$ in the Theorem 1 is satisfied. In Fig. 5 is shown the evolution of the vector $\mu$ and demonstrated that $\hat{\mu} = 0$ guarantee the Condition 2. Thus, it is possible to assert that the dynamic model defined by (5)-(7) satisfies the approximations (26) and (27). The Fig. 4(a) presents the tracking of the trajectory for four different velocities: 2.34 cm/s, 4.68 cm/s, 7.02 cm/s and 9.36 cm/s. For each speed were measured deviations $d$. A better detailing about the deviations associated with both cases ($\epsilon = 0$ and $\epsilon \neq 0$) is presented in Table 1. All these deviations were measured by taking the maximum distance with respect to the first corner of the rhombus. It can be noted that when the speed $v^*_n$ increases the deviations are also larger (see the second column in the Table 1). This is because the increased curvature at corners of the rhombus, i.e., when $R$ is sufficiently larger the contribution of slipping is greater. Then, a higher speed incurs a greater slipping. Theoretically, a model that argues such rate of increase is the model of pseudo-slipping. For more details see (D’Andréa-Novel et al., 1995).

By setting $\epsilon = 0$ the dynamic model becomes rigid. In Fig. 4(b) is shown the behavior of the tracking. However, this behavior presents a greater deviation than the case $\epsilon \neq 0$ (compare the deviations in Table 1).
5 Final remarks

In this paper, the path tracking control problem of a WMR with slipping has been considered. A robust controller based in a nonlinear state feedback for the dynamic model of the WMR also has been proposed. The dynamic model was considered by using the singular perturbations theory (see equations (5)-(7)). The controller was designed in two parts: on the one hand, the kinematic controller was projected by using the curvilinear coordinates and the other hand the dynamic controller essentially based in inverse dynamic (compare (29) and (30)). The control law (30) was used in the dynamic model for the cases when $\varepsilon = 0$ (totally rigid) and when $\varepsilon \neq 0$ (flexible). The results observed in the subsection 4.2 indicates that the consideration of the flexible system is better than the rigid system. However, the deviations observed in the Table 1 can be improved by choosing a minor value of $\delta_1$ or by choosing larger values of $k_2$ and $k_3$.

Acknowledgment

The authors would like to thank to the CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior), to the CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) and to the FAPESP (Fundação de Amparo à Pesquisa do Estado da Bahia) for the support given to this research.

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